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## HARDY INEQUALITY WITH THREE MEASURES ON MONOTONE FUNCTIONS

MARIA JOHANSSON, VLADIMIR D. STEPANOV AND ELENA P. USHAKOVA

(communicated by L.-E. Persson)

*Abstract.* Characterization of  $L^p_v[0, \infty) - L^q_\mu[0, \infty)$  boundedness of the general Hardy operator  $(H_s f)(x) = \left( \int_{[0,x]} f^s u d\lambda \right)^{\frac{1}{s}}$  restricted to monotone functions  $f \geq 0$  for  $0 < p, q, s < \infty$  with positive Borel  $\sigma$ -finite measures  $\lambda, \mu$  and  $\nu$  is obtained.

### 1. Introduction

Let  $\mathfrak{M}^+$  be the class consisting of all Borel functions  $f: [0, \infty) \rightarrow [0, +\infty]$  and  $\mathfrak{M} \downarrow$  ( $\mathfrak{M} \uparrow$ ) be a subclass of  $\mathfrak{M}^+$  which consists of all non-increasing (non-decreasing) functions  $f \in \mathfrak{M}^+$ . Suppose that  $\lambda, \mu$  and  $\nu$  are positive Borel  $\sigma$ -finite measures on  $[0, \infty)$  and  $u, v, w \in \mathfrak{M}^+$  are weight functions.

For  $0 < p, q, s < \infty$  we study the problem when the Hardy inequality of the form

$$\left( \int_{[0,\infty)} (H_s f)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p w d\nu \right)^{\frac{1}{p}}, \quad (1.1)$$

holds for all  $f \in \mathfrak{M} \downarrow$  or for all  $f \in \mathfrak{M} \uparrow$ , where

$$(H_s f)(x) := \left( \int_{[0,x]} f^s u d\lambda \right)^{\frac{1}{s}}. \quad (1.2)$$

Since by the substitution  $f^s \rightarrow f$  the inequality (1.1) can be reduced to the equivalent inequality with new parameters  $p$  and  $q$  of the form

$$\left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p w d\nu \right)^{\frac{1}{p}}, \quad (1.3)$$

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where

$$(Hf)(x) := (H_1f)(x) = \int_{[0,x]} f u d\lambda \quad (1.4)$$

we may and shall restrict our studies to the inequality (1.3). All the characterizations of (1.1) can be easily reproduced from the results for (1.3).

The weighted inequality (1.3) for  $f \in \mathfrak{M} \downarrow$ , when  $\lambda = \mu = \nu$  is the Lebesgue measure, was essentially characterized in [9] and [13] with the complement for the case  $0 < q < 1 = p$  in [12] and recent contribution in [1] for the case  $0 < q < p < 1$ . In fact, [9], [13], [12] and [1] deal with the case  $u(x) = 1$ , but a weight  $u$  can be incorporated with no change in the arguments. A piece of historical remarks and the literature can be found in ([3] and [4], Chapter 6). We summarize these results in the following

**THEOREM 1.1.** *Let  $\lambda = \mu = \nu$  be the Lebesgue measure. Then the inequality (1.3) holds for all  $f \in \mathfrak{M} \downarrow$  if and only if:*

(a)  $1 < p \leq q < \infty$ ,  $\max(A_0, A_1) < \infty$ , where

$$A_0 := \sup_{t>0} \left( \int_0^t \left( \int_0^s u \right)^q \nu(s) ds \right)^{\frac{1}{q}} \left( \int_0^t w \right)^{-\frac{1}{p}},$$

and

$$A_1 := \sup_{t>0} \left( \int_t^\infty \nu \right)^{\frac{1}{q}} \left( \int_0^t \left( \int_0^s u \right)^{p'} \left( \int_0^s w \right)^{-p'} w(s) ds \right)^{\frac{1}{p'}}$$

and  $C \approx A_0 + A_1$ .

(b)  $0 < q < p < \infty$ ,  $1 < p < \infty$ ,  $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$ ,  $\max(B_0, B_1) < \infty$ , where

$$B_0 := \left( \int_0^\infty \left( \int_0^t w \right)^{-\frac{r}{p}} \left( \int_0^t \left( \int_0^s u \right)^q \nu(s) ds \right)^{\frac{r}{p}} \left( \int_0^t u \right)^q \nu(t) dt \right)^{\frac{1}{r}},$$

and

$$B_1 := \left( \int_0^\infty \left( \int_t^\infty \nu \right)^{\frac{r}{p}} \left( \int_0^t \left( \int_0^s u \right)^{p'} \left( \int_0^s w \right)^{-p'} w(s) ds \right)^{\frac{r}{p'}} \nu(t) dt \right)^{\frac{1}{r}}$$

and  $C \approx B_0 + B_1$ .

(c)  $0 < q < p \leq 1$ ,  $\max(B_0, \mathcal{B}_1) < \infty$ , where

$$\mathcal{B}_1 := \left( \int_0^\infty \left( \operatorname{esssup}_{s \in [0,t]} \left( \int_0^s u \right)^p \left( \int_0^s w \right)^{-1} \right)^{\frac{r}{p}} \left( \int_t^\infty \nu \right)^{\frac{r}{p}} \nu(t) dt \right)^{\frac{1}{r}}$$

and  $C \approx B_0 + \mathcal{B}_1$ .

(d)  $0 < p \leq q < \infty$ ,  $0 < p \leq 1$ ,  $\max(A_0, \mathcal{A}_1) < \infty$ , where

$$\mathcal{A}_1 := \sup_{t>0} \left( \int_0^t u \right) \left( \int_t^\infty \nu \right)^{\frac{1}{q}} \left( \int_0^t w \right)^{-\frac{1}{p}}$$

and  $C \approx A_0 + \mathcal{A}_1$ .

It is important to note, that the weighted case of (1.3) for  $1 < p, q < \infty$  was solved in [9] by proving *the principle of duality* which allows to reduce an inequality with a positive operator on monotone functions to an inequality with modified operator on non-negative functions. The other cases, when  $p, q \notin (1, \infty)$  were studied by different methods.

Our aim is twofold. First we study the inequality (1.3) in the case  $0 < p \leq 1$  proving a complete analog of the parts (c) and (d) of Theorem 1.1 (Section 3). In the case  $0 < q < p \leq 1$  our method is based on the characterization of the Hardy inequality on nonnegative functions in the case  $0 < q < 1 = p$ , which we establish in Section 3 (Theorem 3.1). This approach is direct and different from discretization methods of [1] and [2].

Hardy inequality (1.3) on monotone functions with two different measures was recently investigated by G. Sinnamon [11]. Namely, for  $1 < p < \infty$  and  $0 < q < \infty$  the author established the equivalence of (1.3) with  $u \equiv v \equiv w \equiv 1$  and  $d\lambda = dv$  for  $f \in \mathfrak{M}^+$  to the same inequality restricted to  $f \in \mathfrak{M} \downarrow$ . Moreover, such equivalence takes place also for more general operator than (1.4), that is for the operator  $(Kf)(x) = \int_{[0,x]} k(x,y)f(y)d\lambda(y)$  with a kernel  $k(x,y) \geq 0$ , which is monotone in the variable  $y$  (see [5, Theorem 2.3]). Moreover, G. Sinnamon [11] extended the Sawyer principle of duality for measures. We apply this extension to characterize (1.3) in case  $1 < p, q < \infty$  (Section 4) combining with the recent results by D.V. Prokhorov [6] for the inequality (1.3) on  $f \in \mathfrak{M}^+$  with  $1 < p < \infty$  and  $0 < q < \infty$  extended by the same author for the Hardy operator with Oinarov kernel [7].

We use the following notations and conventions.  $A \ll B$  means that  $A \leq cB$  with  $c$  depending only on  $p$  and  $q$ ,  $A \approx B$  is equivalent to  $A \ll B \ll A$ . Uncertainties of the form  $0 \cdot \infty$  are taken to be zero. We also use the notation  $:=$  for introducing new quantities.

## 2. Preliminary remarks

Denote

$$\Lambda_f(x) := \int_{[0,x]} f d\lambda, \quad \text{and} \quad \bar{\Lambda}_f(x) := \int_{[x,\infty)} f d\lambda. \quad (2.1)$$

We need the following statements.

LEMMA 2.1. ([6], Lemma 1) *If  $\gamma > 0$ , then*

$$\frac{\Lambda_f(\infty)^{\gamma+1}}{\max\{1, \gamma+1\}} \leq \int_{[0,\infty)} f(x) \Lambda_f(x)^\gamma d\lambda(x) \leq \frac{\Lambda_f(\infty)^{\gamma+1}}{\min\{1, \gamma+1\}} \quad (2.2)$$

*holds. If  $\gamma \in (-1, 0)$  and  $\Lambda_f(\infty) < +\infty$ , then (2.2) holds.*

LEMMA 2.2. ([6], Lemma 2) *If  $\gamma > 0$ , then*

$$\frac{\bar{\Lambda}_f(0)^{\gamma+1}}{\max\{1, \gamma+1\}} \leq \int_{[0,\infty)} f(x) \bar{\Lambda}_f(x)^\gamma d\lambda(x) \leq \frac{\bar{\Lambda}_f(0)^{\gamma+1}}{\min\{1, \gamma+1\}} \quad (2.3)$$

*holds. If  $\gamma \in (-1, 0)$  and  $\bar{\Lambda}_f(0) < +\infty$ , then (2.3) holds.*

The following two statements can be obtained from [[10], Lemma 1.2] (see also [[11], Proposition 1.5]).

LEMMA 2.3. *Let  $f \in \mathfrak{M} \uparrow$  with  $f(0) = 0$  and let  $\eta$  be a Borel measure on  $[0, \infty)$ . Then there exist  $f_0 \in \mathfrak{M} \uparrow$  and the sequence  $\{h_n\}_{n \geq 1} \subset \mathfrak{M}^+$  such that*

- (1)  $f_0(x) \leq f(x)$  for all  $x \in [0, \infty)$ .
- (2)  $f_0(x) = f(x)$  for  $\eta$ -a.e.  $x \in [0, \infty)$ .
- (3)  $f_n(x) := \int_{[0,x]} h_n d\eta \leq f_0(x)$  for all  $x \in [0, \infty)$ .
- (4) For all  $x \in [0, \infty)$  the sequence  $\{f_n(x)\}_{n \geq 1}$  is nondecreasing in  $n$  and  $f_0(x) = \lim_{n \rightarrow \infty} f_n(x)$   $\eta$ -a.e.  $x \in [0, \infty)$ .

LEMMA 2.4. *Let  $f \in \mathfrak{M} \downarrow$  with  $f(+\infty) = 0$  and let  $\eta$  be a Borel measure on  $[0, \infty)$ . Then there exist  $f_0 \in \mathfrak{M} \downarrow$  and the sequence  $\{h_n\}_{n \geq 1} \subset \mathfrak{M}^+$  such that*

- (1)  $f_0(x) \leq f(x)$  for all  $x \in [0, \infty)$ .
- (2)  $f_0(x) = f(x)$  for  $\eta$ -a.e.  $x \in [0, \infty)$ .
- (3)  $f_n(x) := \int_{[x,\infty)} h_n d\eta \leq f_0(x)$  for all  $x \in [0, \infty)$ .
- (4) For all  $x \in [0, \infty)$  the sequence  $\{f_n(x)\}_{n \geq 1}$  is nondecreasing in  $n$  and  $f_0(x) = \lim_{n \rightarrow \infty} f_n(x)$   $\eta$ -a.e.  $x \in [0, \infty)$ .

REMARK 2.5. Two similar lemmas are valid for the approximation from above.

The following statements are taken from [7] and concern the weighted  $L^p_\lambda[0, \infty) - L^q_\mu[0, \infty)$  inequality with the operator of the form

$$(K_u f)(x) = \int_{[0,x]} k(x,y) u(y) f(y) d\lambda(y).$$

Here the kernel  $k(x,y) \geq 0$  is  $\mu \times \lambda$ -measurable on  $[0, \infty) \times [0, \infty)$  and satisfies the following Oinarov condition. There is a constant  $D \geq 1$  such that

$$D^{-1} k(x,y) \leq k(x,z) + k(z,y) \leq D k(x,y), \quad 0 \leq y \leq z \leq x. \quad (2.4)$$

THEOREM 2.6. *Let  $1 < p \leq q < \infty$ . Then the inequality*

$$\left( \int_{[0,\infty)} (K_u f)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}} \quad (2.5)$$

holds for all  $f \in \mathfrak{M}^+$  if and only if  $\mathbb{A} := \max(\mathbb{A}_{0,1}, \mathbb{A}_{0,2}) < \infty$ , where

$$\mathbb{A}_{0,1} := \sup_{t \in [0,\infty)} \left( \int_{[t,\infty)} v(x) k(x,t)^q d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[0,t]} u^{p'} d\lambda \right)^{\frac{1}{p'}},$$

$$\mathbb{A}_{0,2} := \sup_{t \in [0,\infty)} \left( \int_{[t,\infty)} v d\mu \right)^{\frac{1}{q}} \left( \int_{[0,t]} k(t,y)^{p'} u(y)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}.$$

If  $1 < q < p < \infty$  and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ , then inequality (2.5) holds for all  $f \in \mathfrak{M}^+$  if and only if  $\mathbb{B} := \max(\mathbb{B}_{0,1}, \mathbb{B}_{0,2}) < \infty$ , where

$$\mathbb{B}_{0,1} := \left( \int_{[0,\infty)} \left( \int_{[t,\infty)} v(x) k(x,t)^q d\mu(x) \right)^{\frac{r}{q}} \left( \int_{[0,t]} u^{p'} d\lambda \right)^{\frac{r}{q'}} u(t)^{p'} d\lambda(t) \right)^{\frac{1}{r}},$$

$$\mathbb{B}_{0,2} := \left( \int_{[0,\infty)} \left( \int_{[t,\infty)} v d\mu \right)^{\frac{r}{p}} \left( \int_{[0,t]} k(t,y)^{p'} u(y)^{p'} d\lambda(y) \right)^{\frac{r}{p'}} v(t) d\mu(t) \right)^{\frac{1}{r}}.$$

The next statement is an analog of the previous theorem for the operator  $K_u^*$  of the dual form

$$(K_u^* f)(x) = \int_{[x,\infty)} k(y,x) u(y) f(y) d\lambda(y)$$

with a kernel satisfying Oinarov's condition (2.4).

**THEOREM 2.7.** Let  $1 < p \leq q < \infty$ . Then the inequality

$$\left( \int_{[0,\infty)} (K_u^* f)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}} \quad (2.6)$$

holds for all  $f \in \mathfrak{M}^+$  if and only if  $\mathbb{A}^* := \max(\mathbb{A}_{0,1}^*, \mathbb{A}_{0,2}^*) < \infty$ , where

$$\mathbb{A}_{0,1}^* := \sup_{t \in [0,\infty)} \left( \int_{[0,t]} v(x) k(t,x)^q d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[t,\infty)} u^{p'} d\lambda \right)^{\frac{1}{p'}},$$

$$\mathbb{A}_{0,2}^* := \sup_{t \in [0,\infty)} \left( \int_{[0,t]} v d\mu \right)^{\frac{1}{q}} \left( \int_{[t,\infty)} k(y,t)^{p'} u(y)^{p'} d\lambda(y) \right)^{\frac{1}{p'}}.$$

If  $1 < q < p < \infty$  and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ , then inequality (2.6) holds for all  $f \in \mathfrak{M}^+$  if and only if  $\mathbb{B}^* := \max(\mathbb{B}_{0,1}^*, \mathbb{B}_{0,2}^*) < \infty$ , where

$$\mathbb{B}_{0,1}^* := \left( \int_{[0,\infty)} \left( \int_{[0,t]} v(x) k(t,x)^q d\mu(x) \right)^{\frac{r}{q}} \left( \int_{[t,\infty)} u^{p'} d\lambda \right)^{\frac{r}{q'}} u(t)^{p'} d\lambda(t) \right)^{\frac{1}{r}},$$

$$\mathbb{B}_{0,2}^* := \left( \int_{[0,\infty)} \left( \int_{[0,t]} v d\mu \right)^{\frac{r}{p}} \left( \int_{[t,\infty)} k(y,t)^{p'} u(y)^{p'} d\lambda(y) \right)^{\frac{r}{p'}} v(t) d\mu(t) \right)^{\frac{1}{r}}.$$

In the following theorems we collect weight versions of the results obtained by G. Sinnamon in [11] for embeddings the cones of monotone functions. Put

$$W(t) := \int_{[0,t]} w dv, \quad \text{and} \quad \bar{W}(x) := \int_{[x,\infty)} w dv. \quad (2.7)$$

THEOREM 2.8. *If  $0 < p \leq q < \infty$ , then*

$$\sup_{F \in \mathfrak{M} \downarrow} \frac{\left( \int_{[0, \infty)} F^q v d\mu \right)^{\frac{1}{q}}}{\left( \int_{[0, \infty)} F^p w d\nu \right)^{\frac{1}{p}}} = \sup_{x \in [0, \infty)} \frac{\left( \int_{[0, x]} v d\mu \right)^{\frac{1}{q}}}{\left( \int_{[0, x]} w d\nu \right)^{\frac{1}{p}}}. \quad (2.8)$$

THEOREM 2.9. *If  $0 < q < p < \infty$ , and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$  then*

$$\sup_{F \in \mathfrak{M} \downarrow} \frac{\left( \int_{[0, \infty)} F^q v d\mu \right)^{\frac{1}{q}}}{\left( \int_{[0, \infty)} F^p w d\nu \right)^{\frac{1}{p}}} \approx \left( \int_{[0, \infty)} w(y) \left( \int_{[y, \infty)} W^{-1} v d\mu \right)^{\frac{r}{q}} dv(y) \right)^{\frac{1}{r}}. \quad (2.9)$$

Analogous results take place for  $F \in \mathfrak{M} \uparrow$ .

THEOREM 2.10. *If  $0 < p \leq q < \infty$ , then*

$$\sup_{F \in \mathfrak{M} \uparrow} \frac{\left( \int_{[0, \infty)} F^q v d\mu \right)^{\frac{1}{q}}}{\left( \int_{[0, \infty)} F^p w d\nu \right)^{\frac{1}{p}}} = \sup_{x \in [0, \infty)} \frac{\left( \int_{[x, \infty)} v d\mu \right)^{\frac{1}{q}}}{\left( \int_{[x, \infty)} w d\nu \right)^{\frac{1}{p}}}. \quad (2.10)$$

THEOREM 2.11. *If  $0 < q < p < \infty$  and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ , then*

$$\sup_{F \in \mathfrak{M} \uparrow} \frac{\left( \int_{[0, \infty)} F^q v d\mu \right)^{\frac{1}{q}}}{\left( \int_{[0, \infty)} F^p w d\nu \right)^{\frac{1}{p}}} \approx \left( \int_{[0, \infty)} w(y) \left( \int_{[0, y]} \bar{W}^{-1} v d\mu \right)^{\frac{r}{q}} dv(y) \right)^{\frac{1}{r}}. \quad (2.11)$$

Note that Theorems 2.9 and 2.11 with  $q = 1$  give analogs of Sawyer's principle of duality with general Borel measures.

### 3. The case $0 < p \leq 1$

We need the following extension of ([12], Theorem 3.3) from the weighted case to the case of measures.

THEOREM 3.1. *Let  $0 < q < 1$ ,  $v = v_a + v_s$ , where  $dv_a = \frac{dv_a}{d\lambda} d\lambda$  and  $v_s \perp \lambda$ .*

*Then*

$$\left( \int_{[0, \infty)} \left( \int_{[0, x]} f u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f w d\nu \quad (3.1)$$

*holds for all  $f \in \mathfrak{M}^+$  if and only if*

$$\mathcal{B} := \left( \int_{[0, \infty)} \left( \int_{[0, y]} \frac{v(z) d\mu(z)}{\tilde{w}_\downarrow(z)} \right)^{\frac{q}{1-q}} v(y) d\mu(y) \right)^{\frac{1-q}{q}} < \infty,$$

where

$$\tilde{w} := \frac{w}{u} \frac{dv_a}{d\lambda} \quad \text{and} \quad \tilde{w}(x)_\downarrow := \operatorname{ess\,inf}_{t \in [0, x]} \tilde{w}(t). \quad (3.2)$$

Moreover,  $C \approx \mathcal{B}$ .

*Proof.* Let us start with proving that (3.1) is equivalent to the following inequality

$$\left( \int_{[0, \infty)} \left( \int_{[0, x]} f u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f w \frac{dv_a}{d\lambda} d\lambda. \quad (3.3)$$

Obviously, (3.3) implies (3.1). Let (3.1) hold and  $f \in \mathfrak{M}^+$ . If  $v_s \perp \lambda$ , then there exists  $A \subset [0, \infty)$  such that  $\lambda(A) = 0$ ,  $\operatorname{supp} v_s = A$  and  $\operatorname{supp} v_a = [0, \infty) \setminus A$ . Let  $\tilde{f} = f \chi_{[0, \infty) \setminus A}$ . Then

$$\begin{aligned} \left( \int_{[0, \infty)} \left( \int_{[0, x]} f u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} &= \left( \int_{[0, \infty)} \left( \int_{[0, x]} \tilde{f} u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \\ &\leq C \int_{[0, \infty)} \tilde{f} w dv = C \left( \int_{[0, \infty)} \tilde{f} w dv_a + \int_{[0, \infty)} \tilde{f} w dv_s \right) = C \int_{[0, \infty)} f w dv_a. \end{aligned}$$

Now if we use (3.2), then (3.3) is equivalent to

$$\left( \int_{[0, \infty)} \left( \int_{[0, x]} f u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f u \tilde{w} d\lambda. \quad (3.4)$$

Then, by [10, Theorem 3.1] and changing  $f u$  to  $f$ , we get that (3.4) is equivalent to

$$\left( \int_{[0, \infty)} \left( \int_{[0, x]} f d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} f \tilde{w}_\downarrow d\lambda. \quad (3.5)$$

Now we follow the proof of [12, Theorem 3.3]. First let  $\tilde{w}_\downarrow(x) = \int_{[x, \infty)} b d\lambda$  for  $\lambda$ -a.e.  $x \in [0, \infty)$ ,  $\int_{[0, \infty)} b d\lambda = \infty$  and  $\int_{[x, \infty)} b d\lambda < \infty$ . Then by changing order of integration the right hand side of (3.5) is equal to

$$C \int_{[0, \infty)} \left( \int_{[0, x]} f d\lambda \right) b(x) d\lambda(x)$$

and so (3.5) is equivalent to

$$\left( \int_{[0, \infty)} \left( \int_{[0, x]} f d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} \left( \int_{[0, x]} f d\lambda \right) b(x) d\lambda(x). \quad (3.6)$$

Since  $\int_{[0, x]} f d\lambda$  is increasing we can replace it with  $F$  and so (3.6) is equivalent to

$$\left( \int_{[0, \infty)} F^q v d\mu \right)^{\frac{1}{q}} \leq C \int_{[0, \infty)} F b d\lambda \quad \text{with } F \in \mathfrak{M}^+ \uparrow. \quad (3.7)$$



By [11, Theorem 2.5] and using Lemma 2.2 we get

$$\begin{aligned}
 C &\approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} \frac{v(y) d\mu(y)}{\tilde{w}_\downarrow(y)} \right)^{\frac{1}{1-q}} b(x) d\lambda(x) \right)^{\frac{q}{1-q}} \\
 &\approx \left( \int_{[0,\infty)} \int_{[0,x]} \frac{v(y) d\mu(y)}{\tilde{w}_\downarrow(y)} \left( \int_{[0,y]} \frac{v(z) d\mu(z)}{\tilde{w}_\downarrow(z)} \right)^{\frac{q}{1-q}} b(x) d\lambda(x) \right)^{\frac{1-q}{q}} \\
 &= \left( \int_{[0,\infty)} \left( \int_{[0,y]} \frac{v(z) d\mu(z)}{\tilde{w}_\downarrow(z)} \right)^{\frac{q}{1-q}} v(y) d\mu(y) \right)^{\frac{1-q}{q}}.
 \end{aligned}$$

For a general  $\tilde{w}_\downarrow$  we may and shall suppose that  $\tilde{w}_\downarrow(x) < \infty$  for all  $x > 0$ . Let  $N \in \mathbb{N}$  and

$$w_N(x) := \chi_{[0,N]}(x) \tilde{w}_\downarrow(x).$$

Then  $w_N(+\infty) = 0$  and similar to Lemma 2.4 we find  $w_N^{(0)} \in \mathfrak{M} \downarrow$  and  $h_n \in \mathfrak{M}^+$  ( $n \in \mathbb{N}$ ) such that

- (1)  $w_N(x) \leq w_N^{(0)}(x)$  for all  $x \in [0, \infty)$ .
- (2)  $w_N(x) = w_N^{(0)}(x)$  for  $\lambda$ -a.e.  $x \in [0, \infty)$ .
- (3)  $w_{N,k}(x) := \int_{[x,\infty)} h_k d\lambda \geq w_N^{(0)}(x)$  for all  $x \in [0, \infty)$ .
- (4) The sequence  $\{w_{N,k}(x)\}_{k \geq 1}$  is nonincreasing in  $k$  for all  $x \in [0, \infty)$  and  $w_N^{(0)}(x) = \lim_{k \rightarrow \infty} w_{N,k}(x)$   $\lambda$ -a.e.  $x \in [0, \infty)$ . Then by the previous part of the proof for any  $f \in \mathfrak{M}^+$  we have

$$\begin{aligned}
 &\left( \int_{[0,N]} \left( \int_{[0,x]} f d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \\
 &\ll \left( \int_{[0,N]} \left( \int_{[0,x]} \frac{v(z) d\mu(z)}{w_{N,k}(z)} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} \int_{[0,N]} f w_{N,k} d\lambda.
 \end{aligned}$$

By [6, Lemma 5] this is equivalent to

$$\begin{aligned}
 &\left( \int_{[0,N]} \left( \int_{[0,x]} \frac{f}{w_{N,k}} d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \\
 &\ll \left( \int_{[0,N]} \left( \int_{[0,x]} \frac{v(z) d\mu(z)}{w_{N,k}(z)} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} \int_{[0,N]} f d\lambda.
 \end{aligned}$$

By (3) and (1) we have  $\frac{1}{w_{N,k}(z)} \leq \frac{1}{w_N(z)}$  and by (4), (2) and Monotone Convergence Theorem

$$\lim_{k \rightarrow \infty} \int_{[0,x]} \frac{f}{w_{N,k}} d\lambda = \int_{[0,x]} \frac{f}{w_N^{(0)}} d\lambda = \int_{[0,x]} \frac{f}{w_N} d\lambda.$$

Making the reverse change  $\frac{f}{w_N} \rightarrow f$  we find

$$\begin{aligned} & \left( \int_{[0,N]} \left( \int_{[0,x]} f d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \\ & \ll \left( \int_{[0,N]} \left( \int_{[0,x]} \frac{v(z) d\mu(z)}{w_N(z)} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} \int_{[0,N]} f w_N d\lambda \\ & = \left( \int_{[0,N]} \left( \int_{[0,x]} \frac{v(z) d\mu(z)}{\tilde{w}_\downarrow(z)} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} \int_{[0,N]} f \tilde{w}_\downarrow d\lambda \\ & \leq \mathcal{B} \int_{[0,\infty)} f \tilde{w}_\downarrow d\lambda. \end{aligned}$$

Letting  $N \rightarrow \infty$  we arrive at  $C \ll \mathcal{B}$ . To show the reverse inequality we again approximate  $\tilde{w}_\downarrow$  from above by a monotone sequence of functions  $w_k(x) := \int_{[x,\infty)} b_k d\lambda \downarrow \tilde{w}_\downarrow$ . Then applying (3.6), (3.7) and [11, Theorem 2.5] we find

$$\left( \int_{[0,\infty)} \left( \int_{[0,y]} \frac{v(z) d\mu(z)}{w_k(z)} \right)^{\frac{q}{1-q}} v(y) d\mu(y) \right)^{\frac{1-q}{q}} \ll C$$

and since  $w_k^{-1} \uparrow \tilde{w}_\downarrow^{-1}$  the result follows.  $\square$

**DEFINITION 3.2.** Let  $w \in \mathfrak{M} \downarrow$  and be continuous on the left. It is known ([8, Chapter 12, §3]), that there exists a Borel measure, say  $\eta_w$ , such that  $w(x) = \int_{[x,\infty)} d\eta_w + w(+\infty)$ . We say that  $w \in \mathcal{J}_2(0)$  if there exist a constant  $C \geq 1$  such that

$$\frac{1}{w(x)} - \frac{1}{w(0)} \leq C \int_{[0,x]} \frac{d\eta_w}{w^2}, \quad x > 0.$$

**COROLLARY 3.3.** Let  $0 < q < 1$ ,  $w \in \mathfrak{M} \downarrow$  and  $w \in \mathcal{J}_2(0)$ . Then

$$\left( \int_{[0,\infty)} \left( \int_{[0,x]} h d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} h w d\lambda$$

holds for all  $h \in \mathfrak{M}^+$  if and only if

$$\mathbb{B} := \left( \int_{[0,\infty)} \left( \int_{[0,x]} \frac{v d\mu}{w} \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} < \infty.$$

Moreover,  $C \approx \mathbb{B} \approx \mathbb{B}_0 + \mathbb{B}_1$ , where

$$\mathbb{B}_0 := \left( \int_{[0,\infty)} v d\mu \right)^{\frac{1}{q}} w(0)^{-\frac{1}{p}},$$

$$\mathbb{B}_1 := \left( \int_{[0,\infty)} w(x)^{-\frac{q}{1-q}} \left( \int_{[x,\infty)} v d\mu \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}}.$$

*Proof.* It follows from Theorem 3.1, Lemma 2.2 and [11, Theorem 2.6].  $\square$

Denote

$$\Lambda(t) := \Lambda_u(t) = \int_{[0,t]} u d\lambda \quad (3.8)$$

and observe that by the change  $f^p \rightarrow f$  in the inequality (1.3) we get the following equivalent inequality

$$\left( \int_{[0,\infty)} \left( Hf^{\frac{1}{p}} \right)^q v d\mu \right)^{\frac{p}{q}} \leq C^p \left( \int_{[0,\infty)} f w dv \right), \quad f \in \mathfrak{M} \downarrow. \quad (3.9)$$

**THEOREM 3.4.** (a) Let  $0 < p \leq q < \infty$  and  $0 < p \leq 1$ . Then (1.3) holds for all  $f \in \mathfrak{M} \downarrow$  if and only if

$$A_0 := \sup_{t \in [0,\infty)} \left( \int_{[0,t]} w dv \right)^{-\frac{1}{p}} \left( \int_{[0,t]} \Lambda^q v d\mu \right)^{\frac{1}{q}} < \infty,$$

$$\mathcal{A}_1 := \sup_{t \in [0,\infty)} \Lambda(t) \left( \int_{[0,t]} w dv \right)^{-\frac{1}{p}} \left( \int_{[t,\infty)} v d\mu \right)^{\frac{1}{q}} < \infty$$

and  $C \approx A_0 + \mathcal{A}_1$ .

(b) Let  $0 < q < 1 = p$ . Then (1.3) holds for all  $f \in \mathfrak{M} \downarrow$  if and only if

$$\mathbb{B}_0 := \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{1}{1-q}} dv(y) \right)^{\frac{1-q}{q}} < \infty,$$

$$\mathbb{B}_1 := \left( \int_{[0,\infty)} \left( \int_{[0,x)} \operatorname{ess\,sup}_{s \in [0,t]} \frac{\Lambda(s)}{W(s)} v(t) d\mu(t) \right)^{\frac{q}{1-q}} v(x) d\mu(x) \right)^{\frac{1-q}{q}} < \infty$$

and  $C \approx \mathbb{B}_0 + \mathbb{B}_1$ .

(c) Let  $0 < q < p < 1$ ,  $\mathcal{V}_p(t) := \operatorname{ess\,sup}_{s \in [0,t]} \frac{\Lambda^p(s)}{W(s)}$ . Then (1.3) holds for all  $f \in \mathfrak{M} \downarrow$  if

$$\mathcal{B}_0 := \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{p}{p-q}} dv(y) \right)^{\frac{p-q}{pq}} < \infty,$$

$$\mathcal{B}_1 := \left( \int_{[0,\infty)} \left( \int_{[0,x]} \mathcal{V}_p(t) v(x) d\mu(x) \right)^{\frac{q}{p-q}} v(t) d\mu(t) \right)^{\frac{p-q}{pq}} < \infty$$

and only if  $\mathcal{B}_0 + \mathcal{B}_1 < \infty$ , provided  $\mathcal{V}_p(t)$  is continuous on  $(0, \infty)$  and  $\frac{1}{\mathcal{V}_p(t)} \in \mathcal{I}_2(0)$ . Then  $C \approx \mathcal{B}_0 + \mathcal{B}_1$ .

*Proof.* (a) Since  $f \in \mathfrak{M} \downarrow$ , then  $(H_u f)(x) \geq f(x) \wedge (x)$  and (1.3) implies

$$\left( \int_{[0,\infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p w dv \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M} \downarrow.$$

It is known (see Theorem 2.8) that  $C = A_0$  for  $0 < p \leq q < \infty$ .

Now, if  $f_t = \chi_{[0,t]}$  in (1.3) then

$$C \left( \int_{[0,t]} w dv \right)^{\frac{1}{p}} \geq \left( \int_{[t,\infty)} (H_u f_t)^q v d\mu \right)^{\frac{1}{q}} = \Lambda(t) \left( \int_{[t,\infty)} v d\mu \right)^{\frac{1}{q}},$$

which implies that  $C \geq \mathcal{A}_1$ . Consequently,  $A_0 + \mathcal{A}_1 \leq 2C$ .

For the sufficiency we suppose first that  $f \in \mathfrak{M} \downarrow$ ,  $f(x) = \int_{[x,\infty)} h u d\lambda$  for  $\lambda$ -a.e.  $x \in [0, \infty)$ , where  $h \in \mathfrak{M}^+$  and  $f(x) \geq \int_{[x,\infty)} h u d\lambda$  for all  $x \in [0, \infty)$ . Let  $0 < p < 1$ . We have by Lemma 2.2

$$\begin{aligned} & \int_{[0,x]} \left( \int_{[s,\infty)} h u d\lambda \right) u(s) d\lambda(s) \\ & \approx \int_{[0,x]} \left( \int_{[s,\infty)} \left( \int_{[y,\infty)} h u d\lambda \right)^{p-1} h(y) u(y) d\lambda(y) \right)^{\frac{1}{p}} u(s) d\lambda(s) \\ & \ll \int_{[0,x]} \left( \int_{[s,x]} \left( \int_{[y,\infty)} h u d\lambda \right)^{p-1} h(y) u(y) d\lambda(y) \right)^{\frac{1}{p}} u(s) d\lambda(s) + \Lambda(x) f(x) \\ & \quad \text{[by Minkowski inequality]} \\ & \leq \left( \int_{[0,x]} \left( \int_{[y,\infty)} h u d\lambda \right)^{p-1} h(y) u(y) \Lambda(y)^p d\lambda(y) \right)^{\frac{1}{p}} + \Lambda(x) f(x). \end{aligned} \tag{3.10}$$

Applying (3.10) we obtain

$$\left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \ll \left( \int_{[0,\infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} + J, \tag{3.11}$$

where

$$J := \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[y,\infty)} h u d\lambda \right)^{p-1} h(y) u(y) \Lambda(y)^p d\lambda(y) \right)^{\frac{q}{p}} v(x) d\mu(x) \right)^{\frac{1}{q}}.$$

For the first term on the right hand side of (3.11) by Theorem 2.8 we have

$$\left( \int_{[0,\infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} \leq A_0 \left( \int_{[0,\infty)} f^p w dv \right)^{\frac{1}{p}}. \quad (3.12)$$

For the second term on the right hand side of (3.11) by Minkowski inequality with  $\frac{q}{p} \geq 1$  and Lemma 2.2 we find

$$\begin{aligned} J &\leq \left( \int_{[0,\infty)} \left( \int_{[y,\infty)} h u d\lambda \right)^{p-1} h(y) u(y) \Lambda(y)^p \left( \int_{[y,\infty)} v d\mu \right)^{\frac{p}{q}} d\lambda(y) \right)^{\frac{1}{p}} \\ &\leq \mathcal{A}_1 \left( \int_{[0,\infty)} \left( \int_{[y,\infty)} h u d\lambda \right)^{p-1} h(y) u(y) \left( \int_{[0,y]} w dv \right) d\lambda(y) \right)^{\frac{1}{p}} \\ &\approx \mathcal{A}_1 \left( \int_{[0,\infty)} \left( \int_{[s,\infty)} h u d\lambda \right)^p w(s) dv(s) \right)^{\frac{1}{p}} \leq \mathcal{A}_1 \left( \int_{[0,\infty)} f^p w dv \right)^{\frac{1}{p}} \end{aligned}$$

and the inequality

$$\left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \ll (A_0 + \mathcal{A}_1) \left( \int_{[0,\infty)} f^p w dv \right)^{\frac{1}{p}} \quad (3.13)$$

in this case follows. For an arbitrary  $f \in \mathfrak{M} \downarrow$  without loss of generality we may suppose that  $f(+\infty) = 0$  and find by Lemma 2.4 that  $f_0 \in \mathfrak{M} \downarrow$  and a sequence  $\{h_n\}_{n \geq 1} \subset \mathfrak{M}^+$  such that

- (1)  $f_0(x) \leq f(x)$  for all  $x \in [0, \infty)$ .
- (2)  $f_0(x) = f(x)$  for  $\lambda$ -a.e.  $x \in [0, \infty)$ .
- (3)  $f_n(x) := \int_{[x,\infty)} h_n u d\lambda \leq f_0(x)$  for all  $x \in [0, \infty)$ .
- (4) For all  $x \in [0, \infty)$  the sequence  $\{f_n(x)\}_{n \geq 1}$  is nondecreasing in  $n$  and  $f_0(x) = \lim_{n \rightarrow \infty} f_n(x)$   $\lambda$ -a.e.  $x \in [0, \infty)$ . Then by the Monotone Convergence

Theorem and (3.13), it yields that

$$\begin{aligned} \left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} &\stackrel{(2)}{=} \left( \int_{[0,\infty)} (Hf_0)^q v d\mu \right)^{\frac{1}{q}} \\ &\stackrel{(4)}{=} \lim_{n \rightarrow \infty} \left( \int_{[0,\infty)} (Hf_n)^q v d\mu \right)^{\frac{1}{q}} \stackrel{(3.13)}{\ll} (A_0 + \mathcal{A}_1) \lim_{n \rightarrow \infty} \left( \int_{[0,\infty)} f_n^p w dv \right)^{\frac{1}{p}} \\ &\stackrel{(3)}{\leq} (A_0 + \mathcal{A}_1) \left( \int_{[0,\infty)} f_0^p w dv \right)^{\frac{1}{p}} \stackrel{(1)}{\leq} (A_0 + \mathcal{A}_1) \left( \int_{[0,\infty)} f^p w dv \right)^{\frac{1}{p}} \end{aligned}$$

and the upper bound  $C \ll A_0 + \mathcal{A}_1$  is proved. The case  $p = 1$  is treated by the same method, but even simpler.

(b) Necessity. It follows from the inequality

$$\left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} f w dv, \quad f \in \mathfrak{M} \downarrow, \quad (3.14)$$

that

$$\left( \int_{[0,\infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} f w dv, \quad f \in \mathfrak{M} \downarrow. \quad (3.15)$$

The last inequality is characterized by  $\mathbb{B}_0$  (see Theorem 2.9 with  $p = 1$ .) Hence,  $\mathbb{B}_0 \leq C$ . Now, suppose  $h \in \mathfrak{M}^+$  and  $f(x) = \int_{[x,\infty)} h u d\lambda$ . Then  $f \in \mathfrak{M} \downarrow$  and (3.14) gives

$$\begin{aligned} \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[s,\infty)} h u d\lambda \right) u(s) d\lambda(s) \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \\ \leq C \int_{[0,\infty)} \left( \int_{[s,\infty)} h u d\lambda \right) w(s) dv(s). \end{aligned}$$

This implies

$$\left( \int_{[0,\infty)} \left( \int_{[0,x]} h \Lambda u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} h W u d\lambda.$$

Changing the variable  $h\Lambda u \rightarrow h$  we obtain

$$\left( \int_{[0,\infty)} \left( \int_{[0,x]} h d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} h \frac{W}{\Lambda} d\lambda.$$

The last inequality is characterized by Theorem 3.1. Consequently,  $\mathbb{B}_1 \ll C$ .

Sufficiency. Again, suppose first, that  $f \in \mathfrak{M} \downarrow$ ,  $f(x) = \int_{[x,\infty)} hud\lambda$  for  $\lambda$ -a.e.  $x \in [0, \infty)$ , where  $h \in \mathfrak{M}$  and  $f(x) \geq \int_{[x,\infty)} hud\lambda$  for all  $x \in [0, \infty)$ . Then we have

$$\begin{aligned} \left( \int_{[0,\infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} &= \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[s,\infty)} hud\lambda \right) u(s) d\lambda(s) \right)^q v d\mu \right)^{\frac{1}{q}} \\ &\ll \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[s,x]} hud\lambda \right) u(s) d\lambda(s) \right)^q v d\mu \right)^{\frac{1}{q}} \\ &\quad + \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} hud\lambda \right)^q \Lambda^q(x) v(x) d\mu(x) \right)^{\frac{1}{q}} \\ &\leq \left( \int_{[0,\infty)} \left( \int_{[0,x]} h \Lambda u d\lambda \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} + \left( \int_{[0,\infty)} f^q \Lambda^q v d\mu \right)^{\frac{1}{q}} \end{aligned}$$

[applying Theorem 3.1 and Theorem 2.9]

$$\begin{aligned} &\ll \mathbb{B}_1 \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} hud\lambda \right) w(x) dv(x) \right) + \mathbb{B}_0 \left( \int_{[0,\infty)} f w dv \right) \\ &\leq (\mathbb{B}_0 + \mathbb{B}_1) \int_{[0,\infty)} f w dv. \end{aligned}$$

For an arbitrary  $f \in \mathfrak{M} \downarrow$  we use the arguments from the end of the part (a).

(c) Sufficiency. To prove (3.9) we again, suppose first that  $f \in \mathfrak{M} \downarrow$ ,  $f(x) = \int_{[x,\infty)} hud\lambda$  for  $\lambda$ -a.e.  $x \in [0, \infty)$ , where  $h \in \mathfrak{M}^+$  and  $f(x) \geq \int_{[x,\infty)} hud\lambda$  for all  $x \in [0, \infty)$ . Then, arguing as before and applying Minkowskii's inequality, we find

$$\begin{aligned} &\left( \int_{[0,\infty)} \left( Hf^{\frac{1}{p}} \right)^q v d\mu \right)^{\frac{p}{q}} \\ &= \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[s,\infty)} hud\lambda \right)^{\frac{1}{p}} u(s) d\lambda(s) \right)^q v(x) d\mu(x) \right)^{\frac{p}{q}} \\ &\ll \left( \int_{[0,\infty)} \left( \int_{[0,x]} \left( \int_{[s,x]} hud\lambda \right)^{\frac{1}{p}} u(s) d\lambda(s) \right)^q v(x) d\mu(x) \right)^{\frac{p}{q}} \\ &\quad + \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} hud\lambda \right)^{\frac{q}{p}} \Lambda^q(x) v(x) d\mu(x) \right)^{\frac{p}{q}} \\ &\leq \left( \int_{[0,\infty)} \left( \int_{[0,x]} h \Lambda^p u d\lambda \right)^{\frac{q}{p}} v(x) d\mu(x) \right)^{\frac{p}{q}} + \left( \int_{[0,\infty)} f^{\frac{q}{p}} \Lambda^q v d\mu \right)^{\frac{p}{q}} \end{aligned}$$

applying Theorem 3.1 and Theorem 2.9

$$\begin{aligned} & \ll \mathcal{B}_1^p \left( \int_{[0,\infty)} \left( \int_{[x,\infty]} h u d\lambda \right) w(x) dv(x) \right) + \mathcal{B}_0^p \left( \int_{[0,\infty)} f w dv \right) \\ & \leq (\mathcal{B}_0^p + \mathcal{B}_1^p) \int_{[0,\infty)} f w dv. \end{aligned}$$

For an arbitrary  $f \in \mathfrak{M} \downarrow$  we again use the arguments from the end of the part (a).

Necessity. The inequality  $\mathcal{B}_0 \leq C$  follows by using similar arguments as in the proof of  $A_0 \leq C$  and  $\mathbb{B}_0 \leq C$  in the parts (a) and (b).

For the rest it is sufficient to show that (3.9) implies the inequality  $C \gg \mathcal{B}_1$ .

Suppose for simplicity, that  $\mathcal{V}_p(0) = 0$ . Let

$$g(t) := \max \left\{ 2^m, m \in \mathbb{Z} : 2^m \leq \mathcal{V}_p^{\frac{t}{p}}(t) \right\}$$

and

$$\tau_m := \inf \left\{ y \in [0, \infty) : 2^m \leq \mathcal{V}_p^{\frac{t}{p}}(y) \right\}.$$

Since  $\mathcal{V}_p(t)$  is continuous, then  $\tau_m$  exists for all  $m \in \mathbb{Z}$ ,  $\tau_m \uparrow$  and

$$\frac{\Lambda(\tau_m)^r}{W(\tau_m)^{\frac{r}{p}}} = 2^m = \mathcal{V}_p^{\frac{t}{p}}(\tau_m) \leq \mathcal{V}_p^{\frac{t}{p}}(t) \leq 2^{m+1}, \quad t \in [\tau_m, \tau_{m+1}],$$

$$g(\tau_m) = 2^m, \quad g(s) \leq 2^{m-1} \text{ for all } s \in [0, \tau_m).$$

We note that

$$g(t) = \sum_{m \in \mathbb{Z}} 2^m \chi_{[\tau_m, \tau_{m+1})}(t) \leq \mathcal{V}_p^{\frac{t}{p}}(t) \quad (3.16)$$

and define

$$f(t) := \int_{[t,\infty)} \frac{\left( \int_{[x,\infty)} v d\mu \right)^{\frac{t}{q}}}{W(x)} dg(x).$$

Then  $f \in \mathfrak{M} \downarrow$  and by Lemma 2.2

$$\begin{aligned} \int_{[0,\infty)} f w dv &= \int_{[0,\infty)} \left( \int_{[x,\infty)} v d\mu \right)^{\frac{t}{q}} dg(x) \\ &\approx \int_{[0,\infty)} g(x) \left( \int_{[x,\infty)} v d\mu \right)^{\frac{t}{p}} v(x) d\mu(x) \\ &\leq \int_{[0,\infty)} \mathcal{V}_p^{\frac{t}{p}}(x) \left( \int_{[x,\infty)} v d\mu \right)^{\frac{t}{p}} v(x) d\mu(x) := \mathcal{B}_{2,1}^r. \end{aligned}$$



On the other hand

$$\begin{aligned}
& \left( \int_{[0,\infty)} \left( \int_{[0,x]} f^{\frac{1}{p}}(y) d\Lambda(y) \right)^q v(x) d\mu(x) \right)^{\frac{1}{q}} \\
& \geq \left( \sum_m \int_{[\tau_m, \tau_{m+1})} v(x) \left( \int_{[0, \tau_m]} \left( \int_{[y, \tau_m]} \frac{\left( \int_{[s, \infty)} v d\mu \right)^{\frac{r}{q}}}{W(s)} dg(s) \right)^{\frac{1}{p}} d\Lambda(y) \right)^q d\mu(x) \right)^{\frac{1}{q}} \\
& \geq \left( \sum_m \left( \int_{[\tau_m, \tau_{m+1})} v d\mu \right) \left( \int_{[\tau_m, \infty)} v d\mu \right)^{\frac{r}{p}} \right. \\
& \quad \times \left. \left( W(\tau_m)^{-\frac{1}{p}} \int_{[0, \tau_m]} (g(\tau_m) - g(y))^{\frac{1}{p}} d\Lambda(y) \right)^q \right)^{\frac{1}{q}} \\
& \gg \left( \sum_m \left( \int_{[\tau_m, \tau_{m+1})} v d\mu \right) \left( \int_{[\tau_m, \infty)} v d\mu \right)^{\frac{r}{p}} \left( \frac{2^{\frac{m}{p}} \Lambda(\tau_m)}{W(\tau_m)^{\frac{1}{p}}} \right)^q \right)^{\frac{1}{q}} \\
& \geq \left( \sum_m 2^m \int_{[\tau_m, \tau_{m+1})} \left( \int_{[s, \infty)} v d\mu \right)^{\frac{r}{p}} v(s) d\mu(s) \right)^{\frac{1}{q}} \\
& \gg \left( \int_{[0, \infty)} \mathcal{V}_p^{\frac{r}{p}}(s) \left( \int_{[s, \infty)} v d\mu \right)^{\frac{r}{p}} v(s) d\mu(s) \right)^{\frac{1}{q}} =: \mathcal{B}_{2,1}^{\frac{r}{q}}
\end{aligned}$$

With such  $f(x)$  the inequality (3.9) implies  $C^p \mathcal{B}_{2,1}^r \gg \mathcal{B}_{2,1}^{\frac{pr}{q}} \Rightarrow C \gg \mathcal{B}_{2,1}$ . Now, if we put  $f = \chi_{\{0\}}$  in (3.9), we find that

$$C \geq \left( \int_{[0, \infty)} v d\mu \right)^{\frac{1}{q}} \left( \frac{W(0)}{\Lambda^p(0)} \right)^{-\frac{1}{p}} = \left( \int_{[0, \infty)} v d\mu \right)^{\frac{1}{q}} \left( \frac{1}{\mathcal{V}_p(0)} \right)^{-\frac{1}{p}} =: \mathcal{B}_{2,0}.$$

It follows from Corollary 3.3, that  $\mathcal{B}_{2,1} + \mathcal{B}_{2,0} \gg \mathcal{B}_1$ . Hence,  $C \gg \mathcal{B}_1$  and the proof is complete.  $\square$

In conclusion of this section we give an analog of part (a) of the previous theorem for non-decreasing functions.

**THEOREM 3.5.** *Let  $0 < p \leq q < \infty$  and  $0 < p \leq 1$ . Then, (1.3) holds for all  $f \in \mathfrak{M} \uparrow$  if and only if*

$$\bar{A}_1 := \sup_{t \in [t, \infty)} \left( \int_{[t, \infty)} \Lambda^q(x, t) v(x) d\mu(x) \right)^{\frac{1}{q}} \bar{W}^{-\frac{1}{p}}(t) < \infty,$$

where

$$\Lambda(x, t) := \int_{[t, x]} u d\lambda,$$

and  $C \approx \bar{A}_1$ .

*Proof.* Replacing  $f$  in (1.3) by  $f_t := \chi_{[t, \infty)}$  we find  $\bar{A}_1 \leq C$ . For sufficiency we suppose that

$$f(x) = \int_{[0, x]} h u d\lambda, \quad h \in \mathfrak{M}^+$$

and let  $0 < p < 1$ . Then, by Minkowskii inequality and Lemma 2.1, we find

$$\begin{aligned} & \int_{[0, x]} \left( \int_{[0, s]} h u d\lambda \right) u(s) d\lambda(s) \\ & \approx \int_{[0, x]} \left( \int_{[0, s]} \left( \int_{[0, y]} h u d\lambda \right)^{p-1} h(y) u(y) d\lambda(y) \right)^{\frac{1}{p}} u(s) d\lambda(s) \\ & \leq \left( \int_{[0, x]} \left( \int_{[0, y]} h u d\lambda \right)^{p-1} h(y) u(y) \Lambda^p(x, y) d\lambda(y) \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, again by Minkowskii inequality

$$\begin{aligned} & \left( \int_{[0, \infty)} (Hf)^q v d\mu \right)^{\frac{1}{q}} \\ & \leq \left( \int_{[0, \infty)} \left( \int_{[0, x]} \left( \int_{[0, y]} h u d\lambda \right)^{p-1} h(y) u(y) \Lambda^p(x, y) d\lambda(y) \right)^{\frac{q}{p}} v(x) d\mu(x) \right)^{\frac{1}{q}} \\ & \leq \left( \int_{[0, \infty)} \left( \int_{[0, y]} h u d\lambda \right)^{p-1} h(y) u(y) \left( \int_{[y, \infty)} \Lambda^q(x, y) v(x) d\mu(x) \right)^{\frac{p}{q}} d\lambda(y) \right)^{\frac{1}{p}} \\ & \leq \bar{A}_1 \left( \int_{[0, \infty)} \left( \int_{[0, y]} h u d\lambda \right)^{p-1} h(y) u(y) \left( \int_{[y, \infty)} w dv \right) d\lambda(y) \right)^{\frac{1}{p}} \\ & \approx \bar{A}_1 \left( \int_{[0, \infty)} f^p w dv \right)^{\frac{1}{p}}. \end{aligned}$$

□

A general case  $f \in \mathfrak{M} \uparrow$  follows by Lemma 2.3 similar to the proof of Theorem 3.4.

#### 4. The case $1 < p, q < \infty$

The result of this section is based on the following statement, which follows from Theorems 2.9 and 2.11 with  $q = 1$ .

**COROLLARY 4.1.** *Let  $(Tf)(x) = \int_{[0,\infty)} k(x,y)f(y)u(y)d\lambda(y)$ , where  $k(x,y)$  is a defined on  $[0,\infty) \times [0,\infty)$ , non-negative,  $\mu \times \lambda$ -measurable kernel.*

(a) *The inequality*

$$\left( \int_{[0,\infty)} (Tf)^q v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p w dv \right)^{\frac{1}{p}} \quad (4.1)$$

for  $f \in \mathfrak{M} \downarrow$ , holds if and only if the inequality

$$\begin{aligned} & \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1}(T^*g)u d\lambda \right)^{p'} dv(y) \right)^{\frac{1}{p'}} \\ & \leq C \left( \int_{[0,\infty)} q^{q'} v d\mu \right)^{\frac{1}{q'}}, \quad g \in \mathfrak{M}^+, \end{aligned} \quad (4.2)$$

holds with  $(T^*g)(z) = \int_{[0,\infty)} k(z,x)g(x)v(x)d\mu(x)$ .

(b) *The inequality (4.1) for  $f \in \mathfrak{M} \uparrow$  holds if and only if the following inequality holds:*

$$\left( \int_{[0,\infty)} w(y) \left( \int_{[0,y]} \bar{W}^{-1}(T^*g)u d\lambda \right)^{p'} dv(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} q^{q'} v d\mu \right)^{\frac{1}{q'}}, \quad g \in \mathfrak{M}^+.$$

Now let us present our result for the case  $1 < p, q < \infty$ .

**THEOREM 4.2.** *Let  $\mathbf{k}(x,y) = \int_{[y,x]} W^{-1}u d\lambda$  and  $f \in \mathfrak{M} \downarrow$ . The inequality (1.3) holds for  $1 < p \leq q < \infty$  if and only if  $\mathcal{A} = \max \{ \mathcal{A}_{0,1} + \mathcal{A}_{0,2} \} < \infty$ , where*

$$\begin{aligned} \mathcal{A}_{0,1} &:= \sup_{t \in [0,\infty)} \left( \int_{[0,t]} w(y) \mathbf{k}(t,y)^{p'} dv(y) \right)^{\frac{1}{p'}} \left( \int_{[t,\infty)} v d\mu \right)^{\frac{1}{q}}, \\ \mathcal{A}_{0,2} &:= \sup_{t \in [0,\infty)} \left( \int_{[0,t]} w dv \right)^{\frac{1}{p'}} \left( \int_{[t,\infty)} v(x) \mathbf{k}(x,t)^q d\mu(x) \right)^{\frac{1}{q}}. \end{aligned}$$

Moreover, if  $C$  is the best constant in (1.3), then  $C = \mathcal{A}$ .

In the case  $1 < q < p < \infty$  the inequality (1.3) holds if and only if  $\mathcal{B} = \max \{ \mathcal{B}_{0,1} + \mathcal{B}_{0,2} \} < \infty$ , where

$$\begin{aligned}\mathcal{B}_{0,1} &= \left( \int_{[0,\infty)} \left( \int_{[0,t]} w(y) \mathbf{k}(t,y)^{p'} dv(y) \right)^{\frac{r}{p'}} \left( \int_{[t,\infty)} v d\mu \right)^{\frac{r}{p}} v(t) d\mu(t) \right)^{\frac{1}{r}}, \\ \mathcal{B}_{0,2} &= \left( \int_{[0,\infty)} \left( \int_{[0,t]} w dv \right)^{\frac{r}{q'}} \left( \int_{[t,\infty)} v(x) \mathbf{k}(x,t)^q d\mu(x) \right)^{\frac{r}{q}} w(t) dv(t) \right)^{\frac{1}{r}}\end{aligned}$$

and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ . Moreover, if  $C$  is the best constant in (1.3), then  $C = \mathcal{B}$ .

*Proof.* Because of Corollary 4.1 (a) the inequality (1.3) is equivalent to

$$\begin{aligned} \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W(x)^{-1} \left( \int_{[x,\infty)} g v d\mu \right) u(x) d\lambda(x) \right)^{p'} dv(y) \right)^{\frac{1}{p'}} \\ \leq C \left( \int_{[0,\infty)} q^{q'} v d\mu \right)^{\frac{1}{q'}}, \quad g \in \mathfrak{M}^+. \end{aligned} \quad (4.3)$$

By changing the order of integration in the left hand side of (4.3) we obtain the Hardy inequality with Oinarov kernel of the form

$$\left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} g(z) \mathbf{k}(z,y) v(z) d\mu(z) \right)^{p'} dv(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} q^{q'} v d\mu \right)^{\frac{1}{q'}}.$$

By substitution  $f = g^{q'}$  and according to Lemma 7 from [7] the last inequality is equivalent to

$$\left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} f(z) \mathbf{k}(z,y) v(z)^{1/q} d\mu(z) \right)^{p'} dv(y) \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} f^{q'} d\mu \right)^{\frac{1}{q'}}.$$

Thus the proof follows by applying Theorem 2.7.  $\square$

Similarly we can obtain the result for non-decreasing functions as follows.

**THEOREM 4.3.** Let  $\bar{\mathbf{k}}(y,x) = \int_{[x,y]} \bar{W}^{-1} u d\lambda$  and  $f \in \mathfrak{M} \uparrow$ . The inequality (1.3) holds for  $1 < p \leq q < \infty$  if and only if  $\bar{\mathcal{A}} = \max \{ \bar{\mathcal{A}}_{0,1} + \bar{\mathcal{A}}_{0,2} \} < \infty$ , where

$$\bar{\mathcal{A}}_{0,1} := \sup_{t \in [0,\infty)} \left( \int_{[t,\infty)} w(y) \bar{\mathbf{k}}(y,t)^{p'} dv(y) \right)^{\frac{1}{p'}} \left( \int_{[0,t]} v d\mu \right)^{\frac{1}{q}},$$

$$\bar{\mathcal{A}}_{0,2} := \sup_{t \in [0, \infty)} \left( \int_{[t, \infty)} w dv \right)^{\frac{1}{p'}} \left( \int_{[0, t]} v(x) \bar{\mathbf{k}}(t, x)^q d\mu(x) \right)^{\frac{1}{q}}.$$

Moreover, if  $C$  is the best constant in (1.3), then  $C = \bar{\mathcal{A}}$ . In the case  $1 < q < p < \infty$  the inequality (1.3) holds if and only if  $\bar{\mathcal{B}} = \max \{ \bar{\mathcal{B}}_{0,1} + \bar{\mathcal{B}}_{0,2} \} < \infty$ , where

$$\bar{\mathcal{B}}_{0,1} := \left( \int_{[0, \infty)} \left( \int_{[t, \infty)} w(y) \bar{\mathbf{k}}(y, t)^{p'} dv(y) \right)^{\frac{r}{p'}} \left( \int_{[0, t]} v d\mu \right)^{\frac{r}{p}} v(t) d\mu(t) \right)^{\frac{1}{r}},$$

$$\bar{\mathcal{B}}_{0,2} := \left( \int_{[0, \infty)} \left( \int_{[t, \infty)} w dv \right)^{\frac{r}{q'}} \left( \int_{[0, t]} v(x) \bar{\mathbf{k}}(t, x)^q d\mu(x) \right)^{\frac{r}{q}} w(t) dv(t) \right)^{\frac{1}{r}}$$

and  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ . Moreover, if  $C$  is the best constant in (1.3), then  $C = \bar{\mathcal{B}}^*$ .

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*Maria Johansson*  
*Department of Mathematics*  
*Luleå University of Technology*  
*SE-97187 Luleå*  
*SWEDEN*  
*e-mail: maria.l.johansson@ltu.se*

*Vladimir D. Stepanov*  
*Department of Mathematical Analysis and Function Theory*  
*Peoples Friendship University*  
*117198 Moscow*  
*RUSSIA*  
*e-mail: vstepanov@sci.pfu.edu.ru*

*Elena P. Ushakova*  
*Computing Centre of Far Eastern Branch*  
*of Russian Academy of Sciences*  
*680000 Khabarovsk*  
*RUSSIA*

Current address:  
*Department of Mathematics*  
*Uppsåla University*  
*SE-751 06 Uppsåla*  
*SWEDEN*  
*e-mail: eleush@sm.luth.se*